Primitive Solar Nebula and Origin of Planets, edited by H. Oya, pp. 79–88.

Shear Instability of the Solar Nebula

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1. Introduction

From the works by Papaloizou and Pringle (1984; 1985), shear instability is considered to occur generally in the accretion disks. They investigated the stability of equilibrium tori for which the rotation law is

$$\Omega_0 \propto r^{-q}, \tag{1}$$

where Ω_0 is the unperturbed angular velocity and r is the distance from the rotation axis. In the case of a slender torus, they, as well as Goldreich, Goodman and Narayan (1986) found that as long as $q > \sqrt{3}$ there is a non-axisymmetric instability. Furthermore, in the case of thin and wide disks, Goldreich and Narayan (1985) showed the existence of unstable acoustic modes.

If shear instability is common in differentially rotating disks, we should investigate whether a model of the solar nebula is stable or not. In the case of the solar nebula, the rotation law is close to the Keplerian, *i.e.* $q \sim 1.5$

and therefore, it has been believed that shear instability does not occur in the solar nebula because $q < \sqrt{3}$. Hanawa (1987), however, showed that there is an instability in the disk where the rotation law $q = 1.5$. In the case of incompressible disks with $q > 1.5$, Sekiya and Miyama (1988) also found an unstable mode. Thus, shear instability is common in differentially rotating systems, the growth rate being greater in the case where $q > \sqrt{3}$ than for $q < \sqrt{3}$.

Since, the theory of the formation of the solar system is based on the solar nebula model (Safronov, 1969; Weidenschilling, 1977; Hayashi, 1981), the stability of the solar nebula is one of the most important problems in the theory of planetary formation. In this paper, we make a preliminary calculation of this problem.

The stability of Hayashi's solar nebula model is examined in §2.

2. The stability of Hayashi's solar nebula model

2.1. *Model of the Solar Nebula*

Here, we review Hayashi's solar nebula model (1981) which will be used in the present study of the stability problem. Hereafter, we use the cylindrical coordinates (r, ϕ, z), where the Sun is located at the origin and the rotation axis of the nebula is assigned to the z-axis. We consider the stage at which almost all dust grains have accumulated into large bodies so that the nebula is transparent to visible and infrared radiations. Then, for a thin disk, the temperature distribution is given by

$$T = \left(\frac{L}{16\pi\sigma r^2}\right)^{\frac{1}{4}} = 280 r^{-\frac{1}{2}} \left(\frac{L}{L_\odot}\right)^{\frac{1}{4}} \quad \mathrm{K}, \tag{2}$$

where we used AU($= 1.5 \times 10^{13}$ cm) as the unit of distance. Further, σ is the Stefan-Boltzmann constant; L and $L_\odot$ are the solar luminosities at the stage considered and at present, respectively.

The surface density distribution is given by

$$\rho_s = 1.7 \times 10^3 r^{-1.5} \quad \mathrm{gcm}^{-2}. \tag{3}$$

Then, the equilibrium density profile of the gaseous nebula is determined by

$$\rho(r,z) = \rho_E r^{-2.75} \exp\left(-\frac{z^2}{z_0^2}\right), \tag{4}$$

where ρ_E is the density at the Earth region. The scaleheight z_0 is given by

$$z_0 = 0.0472\, r^{5/4}, \tag{5}$$

where we assume $L = L_\odot$.

Since it is difficult to deal with the three-dimensional stability problem, we simplify the problem to two-dimensions. That is, we integrate the quantities along the direction of the rotation axis, z, and consider a two-dimensional model (The relation between two-dimensional and three-dimensional disks is discussed by Goldreich, Goodman and Narayan (1986)). Then, the density is replaced by the surface density. In this approximation, we neglect the z-component of the velocity and the z-dependence of the physical quantities, *i.e.*, $v_z = 0$ and $\partial/\partial z = 0$. This simplification is a good approximation of the three-dimensional disks in the case of the solar nebula. In the following, we summarize the unperturbed state of the solar nebula:

1. The gravitational potential is given by $\Psi = -1/r$. In this section, we use the unit as $GM_\odot = 1$, where $M_\odot$ is the solar mass. The self-gravity of the nebula is neglected.
2. The unperturbed sound velocity is given by

$$c_0 = c_E r^{-\alpha}, \tag{6}$$

where $\alpha = 1/4$ and c_E is equal to 0.040 in the case where the temperature is given by Eq. (2). Note that the unit of velocity is the

Keplerian velocity at 1AU. In the present work, we make most of the computations for $c_E = 0.40$, *i.e.*, ten times as large as the realistic value, in order to save on computational time, and only the principal mode is calculated in the realistic case where $c_E = 0.040$.

3. The unperturbed density is given by

$$\rho_0 = \rho_E r^{-\beta}, \tag{7}$$

where $\beta = 1.5$. The explicit value of ρ_E is unnecessary for the following calculations.

4. The unperturbed pressure is given by

$$P_0 = \rho_0 c_0^2/\gamma, \tag{8}$$

where γ is the adiabatic coefficient.

5. In the case of the solar nebula, the main components are hydrogen molecules and helium atoms, and we have $\gamma = 1.43$.
6. The equilibrium angular velocity distribution is given by

$$\Omega_0 = \sqrt{\frac{1}{r^3} - \frac{c_E^2}{\gamma}(2\alpha + \beta) r^{-2\alpha-2}}. \tag{9}$$

7. In the case of the solar nebula, the pressure at the boundary may be constant, since the nebula gets the wind pressure or coronal pressure at its inner boundary surface and the ram pressure of accreting matter at its outer surface.

2.2. *Linear perturbation equations*

Pressure, density and velocities are expanded around the unperturbed state

$$\begin{cases} \rho = & \rho_0 + & \epsilon\rho_1 + \cdots, \\ P = & P_0 + & \epsilon P_1 + \cdots, \\ v_r = & & \epsilon v_{r1} + \cdots, \\ v_\phi = & r\Omega_0 + & \epsilon v_{\phi 1} + \cdots. \end{cases} \tag{10}$$

In order to consider a normal mode analysis, we assume that the linear perturbations have the form $f_1(r)\exp(i\omega t + im\phi)$, where ω $(= \omega_R + i\omega_I)$ is a complex number, t is time and m is an integer. Then the linear perturbation equations are

$$i\bar{\omega}v_{r1} - 2\Omega_0 v_{\phi 1} = -\frac{1}{\rho_0}\frac{dP_1}{dr} + \frac{\rho_1}{\rho_0^2}\frac{dP_0}{dr}, \tag{11}$$

$$i\bar{\omega}v_{\phi 1} + \frac{1}{r}\frac{dl}{dr}v_{r1} = -\frac{imP_1}{\rho_0 r}, \tag{12}$$

$$i\bar{\omega}\rho_1 + \frac{1}{r}\frac{d}{dr}(r\rho_0 v_{r1}) + \frac{im\rho_0}{r}v_{\phi 1} = 0, \tag{13}$$

where

$$\bar{\omega} = \omega + m\Omega_0, \tag{14}$$

$$l = r^2\Omega_0. \tag{15}$$

We solve for v_{r1} and $v_{\phi 1}$ using Eqs. (11) and (12) and obtain

$$v_{r1} = \frac{i}{D_1}\left[\frac{\bar{\omega}}{\rho_0}(\frac{dP_1}{dr} - \rho_1 g_{eff}) + \frac{2m\Omega_0}{r}\frac{P_1}{\rho_0}\right], \tag{16}$$

$$v_{\phi 1} = -\frac{1}{D_1}\left[\frac{1}{r\rho_0}\frac{dl}{dr}(\frac{dP_1}{dr} - \rho_1 g_{eff}) + \frac{m\bar{\omega}}{r}\frac{P_1}{\rho_0}\right], \tag{17}$$

where

$$D_1 = \bar{\omega}^2 - \frac{2\Omega_0}{r}\frac{dl}{dr} = \bar{\omega}^2 - \kappa^2, \tag{18}$$

and

$$g_{eff} = r\Omega_0^2 - \frac{1}{r^2}, \tag{19}$$

and κ is the epicyclic frequency ($\kappa^2 = (2\Omega_0/r)dl/dr$).

Here we consider an adiabatic perturbation of the form

$$\frac{\Delta P}{P_0} = \gamma\frac{\Delta\rho}{\rho_0}, \tag{20}$$

where Δ denotes the Lagrange change of the quantity and γ is the adiabatic coefficient. In this case, this relation can be rewritten as

$$P_1 + \frac{v_{r1}}{i\bar{\omega}}\frac{dP_0}{dr} = c_0^2(\rho_1 + \frac{v_{r1}}{i\bar{\omega}}\frac{d\rho_0}{dr}). \tag{21}$$

These equations can be reduced to one equation, using the quantity, $\chi_1 \equiv P_1/\rho_0$, i.e., the enthalpy perturbation,

$$\frac{d^2\chi_1}{dr^2} + \left(\frac{g_{eff}}{c_0^2} + \frac{1}{r} - \frac{d}{dr}\ln D - \Psi_r\right)\frac{d\chi_1}{dr}$$

$$+\left[-\frac{m^2}{r^2}+\frac{2m\Omega_0}{r\bar{\omega}}\frac{d}{dr}\ln\left(\frac{\Omega_0}{D}\right)+\frac{1}{c_0^2}\left(\frac{2m\Omega_0 g_{eff}}{r\bar{\omega}}+D\right)\right. \quad (22)$$

$$\left.+\Psi_r\left(-\frac{g_{eff}}{c_0^2}-\frac{1}{r}+\frac{d}{dr}\ln D+\frac{2m\Omega_0}{r\bar{\omega}}-\frac{m^2 g_{eff}}{r^2\bar{\omega}^2}\right)-\frac{d\Psi_r}{dr}\right]\chi_1=0,$$

where

$$\Psi_r \equiv -N^2/g_{eff}, \quad (23)$$

$$D \equiv \bar{\omega}^2-\kappa^2-N^2, \quad (24)$$

and

$$N \equiv \sqrt{\frac{1}{\rho_0}\frac{dP_0}{dr}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dr}-\frac{1}{\gamma}\frac{1}{P_0}\frac{dP_0}{dr}\right)}. \quad (25)$$

Here Ψ_r is a quantity proportional to the entropy gradient and N is the local Brunt-Väisälä frequency.

We call the point where the pattern angular velocity, $-\omega_R/m$, coincides with the flow angular velocity Ω_0 the co-rotation point. Note that if ω is real, Eq. (22) has a singularity at this point where $\bar{\omega}=0$. When the system is unstable, however, the co-rotation point is no longer a singular point and according to Lin's rule (1945) we can integrate Eq. (22) along the real axis.

2.3. *Boundary conditions*

In this paper, we consider the cases of constant pressure boundary conditions at the inner and outer boundaries. This boundary condition is represented as

$$\Delta P = P_1 + \frac{dP_0}{dr}\frac{v_{r1}}{i\bar{\omega}} = 0, \quad \text{at} \quad r = r_\pm. \tag{26}$$

We can easily rewrite these conditions using Eqs. (16) and (21) as

$$\frac{d\chi_1}{dr} + \left(\frac{D}{g_{eff}} + \frac{2m\Omega_0}{r\bar{\omega}} - \Psi_r\right)\chi_1 = 0, \tag{27}$$

at the unperturbed radii of the outer and inner boundaries, r_+ and r_-, respectively.

2.4. *The axisymmetric stability*

In the case of stability for axisymmetric perturbations, there is the Solberg and Høiland criterion for stability (Tassoul, 1978) written as

$$\kappa^2 + N^2 \geq 0. \tag{28}$$

Hence we must first check the stability of the model of the solar nebula using this condition. In the case of isentropic disks, i.e., $N = 0$, this stability condition assures that the disks are stable for $q \leq 2$, called the Rayleigh criterion.

2.5. *Numerical Results and Discussion*

In order to check our numerical code, we made a calculation using Hanawa's model (1987) where $\alpha = 1/2$, $\beta = 3/2$ and $\gamma = 5/3$, with rigid boundary conditions and got the same results as Hanawa.

The growth rate ω_I of unstable modes as a function of the outer radii r_+ in the case where $c_E = 0.40$ (this value is ten times as large as the realistic one) and $m = 5$ is shown in Fig. 1, where only modes with co-rotation radius $r_c = 0.86$AU are shown. In Fig. 2, the growth rate of the principal mode in the realistic case where $c_E = 0.04$ and $m = 5$ is shown. From these figures, we see that the system is unstable in some very narrow bands of the outer boundary radius r_+. Thus, Hayashi's solar nebula model (1981)

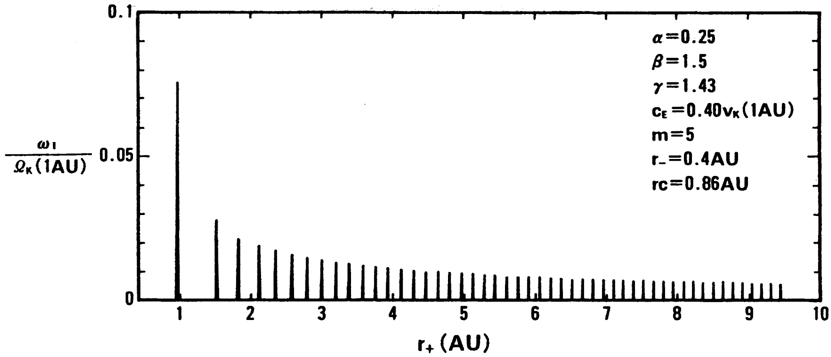

Fig. 1. Growth rate of the unstable mode as a function of the outer boundary radius r_+ for a solar nebula model, in the case where $c_E = 0.40$ (this value is ten times as large as the realistic one), $m = 5$, $r_- = 0.4$AU. Only modes with the co-rotation radius at 0.86AU are shown.

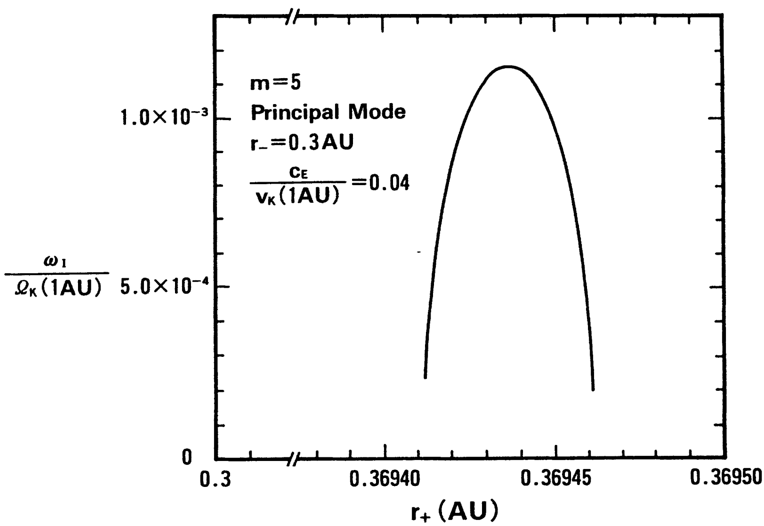

Fig. 2. Growth rate of the principal unstable mode as a function of the outer boundary radius r_+ for a solar nebula model, in the realistic case where $c_E = 0.04$, $m = 5$, $r_- = 0.3$AU.

is possibly stable, however, it is very difficult to terminology show this. In order to do it, we have to carry out another approach, for example, the variational method.

Acknowledgements

This work was supported by the Grant-in-Aid for Scientific Research on Priority Areas (Origin of the Solar System) of Japanese Ministry of Education, Science, and Culture (Nos. 62611002, 63611002, and 01611002).

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